

## Kinky strings in $AdS_5 \times S^5$

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**ABSTRACT:** We construct a family of closed string solutions with kinks in a subspace of  $AdS_5 \times S^5$  and study their properties. In certain limits these solutions become folded pulsating strings, although in general they are made of multiple pulsating rectangles. One unusual feature of these solutions is that their monodromy matrices are trivial, leading to vanishing quasi-momenta. Exact Bäcklund transformations of these solutions are found, again giving vanishing higher conserved charges. We also consider the fluctuation modes around these solutions as well as the semiclassical splitting of these strings.

**KEYWORDS:** Gauge-gravity correspondence, AdS-CFT Correspondence.

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## 1. Introduction

In  $D$ -dimensional flat space the general solution to the bosonic string equations of motion is specified by two unit modulus vectors  $\mathbf{a}^{(1)}$  and  $\mathbf{b}^{(1)}$ , which are the first derivatives w.r.t. the worldsheet left moving and right moving coordinates. Being first derivatives,  $\mathbf{a}^{(1)}$  and  $\mathbf{b}^{(1)}$  do not have to be continuous. When they have discontinuities, the string profile is still continuous, although now with kinks. A kink is the joining point of two string segments where the angle between the two segments is nonzero. People have also considered strings with spikes (sometimes also called cusps), with the spikes being the joining points of two anti-parallel string segments. Both kinky strings and spiky strings in flat space have been investigated quite extensively in the cosmic string context, see for example the review [1]. One important difference between kinky strings and spiky strings is that, while the vectors  $\mathbf{a}^{(1)}$  and  $\mathbf{b}^{(1)}$  are discontinuous functions of worldsheet coordinates for the kinky strings, they are continuous functions for the spiky strings (spikes develop when  $\mathbf{a}^{(1)}$  and  $-\mathbf{b}^{(1)}$ , which “rotate” with different frequencies, cross each other). We shall see this in more detail in section 2. In [2, 3] the spiky strings in flat space were generalized to  $AdS_5 \times S^5$  and their role in the AdS/CFT correspondence was investigated. It is natural to ask whether similar things can be done for the kinky strings, which provides motivation for the current paper.

We shall generalize the kinky strings to the  $R_t \times S^5$  sector of  $AdS_5 \times S^5$  and study their properties such as higher conserved charges, stability under fluctuations, and semi-classical splitting. Of course the study of semi-classical strings in  $AdS_5 \times S^5$  is of particular interest as it may shed light on the  $AdS/CFT$  duality away from the BPS limit and there has been a great deal of progress in understanding semi-classical strings with large quantum numbers such as angular momentum or spin [2, 4–10] (see [11] for a review and a more complete set of references). Those semi-classical strings were mapped to dual “long” gauge theory operators and string energies matched to the operator anomalous dimensions. It would be interesting to study the operators dual to the kinky strings however it is at present unclear to us what these operators should be and so this side of the duality will be left to a future work.

One of the most remarkable facets of the  $\mathcal{N} = 4$  —  $AdS_5 \times S^5$  duality is that both sides appear to be integrable with the gauge theory being described by an integrable spin chain [12, 13]. The string theory has been shown to be classically integrable [14, 15] and there exists some evidence that this persists quantum mechanically [16]. One consequence of integrability is the existence of an infinite number of conserved charges and in general there are several methods to construct these charges. In particular we will concern ourselves with the Lax method involving the monodromy matrix, which has previously been considered for the string theory in [17–21], and the method involving the Bäcklund transformations [22–26]. One peculiar feature of the kinky solutions is that their monodromy matrices are trivial, leading to vanishing quasi momenta. We also solve the Bäcklund transformation for these solutions in an exact manner and show that the associated higher charges all vanish, which is consistent with the finding of trivial monodromies in light of the relation between the monodromy and Bäcklund charges explained in [24]. This feature is in contrast with the quite general constructions made previously in the literature (e.g. [17–21]) where the quasi momentum has a pole as the spectral parameter approaches to  $\pm 1$  and is certainly not vanishing. The reason for this apparent conflict is that the string embedding can have discontinuities in its first worldsheet derivatives, i.e. the string profile can have kinks. Section 3.3 explains this issue in detail. Semi-classical analysis of string decay in both flat spacetime and  $AdS_5 \times S^5$  has been performed in [27–31], and [32], and we carry out this analysis for the splitting of the kinky strings.

This paper is organized as follows. In section 2 we review classical bosonic string solutions with spikes and those with kinks in flat spacetime. In section 3 we then generalize these kinky solutions to the  $R_t \times S^5$  sector of  $AdS_5 \times S^5$ . More specifically, we construct a class of solutions with kinks to the equations of motion, find their Bäcklund transformations as well as their monodromy matrices, and also give some explicit example solutions. In section 4 we compare our solutions with the pulsating strings obtained via the “2d-dual” version of Neumann-Rosochatius(NR) system. In section 5 we investigate the solutions’ stability under small fluctuations. In section 6 we carry out a semi-classical analysis of the splitting of these strings. We end with section 7 discussing the connection of the kinky strings to the  $2d$  flat space strings with point-like longitudinal degrees of freedom [33–36], boost of the kinky solutions to give them angular momenta on  $S^5$  and comparison to the “giant magnons” constructed in [37], as well as other interesting flat space solutions with

kinks whose  $AdS_5 \times S^5$  generalizations are still to be found.

## 2. String solutions in flat space

In this section we review classical string solutions in flat space following mainly the review article [1]. The target space is taken to be  $D$ -dimensional Minkowski

$$ds^2 = \eta_{\mu\nu} dX^\mu dX^\nu; \quad \eta_{\mu\nu} = (-1, 1, \dots, 1) \quad (2.1)$$

with  $\mu, \nu = 0, 1, \dots, D$ . The Nambu-Goto action of the string is

$$S = -T \int d\tau d\sigma \sqrt{-\det h_{\alpha\beta}}; \quad h_{\alpha\beta} = \eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \quad (2.2)$$

with  $\alpha, \beta = 0, 1$  being worldsheet indices. We use the conformal gauge

$$\begin{aligned} \eta_{\mu\nu} (\partial_\tau X^\mu \partial_\tau X^\nu + \partial_\sigma X^\mu \partial_\sigma X^\nu) &= 0 \\ \eta_{\mu\nu} \partial_\tau X^\mu \partial_\sigma X^\nu &= 0 \end{aligned} \quad (2.3)$$

which as usual is written as

$$\begin{aligned} \dot{X}^2 + X'^2 &= 0 \\ \dot{X} \cdot X' &= 0 \end{aligned} \quad (2.4)$$

and in the conformal gauge the equation of motion is

$$\ddot{X}^\mu - X''^\mu = 0 \quad (2.5)$$

The conformal gauge still leaves some freedom which we further restrict by setting  $t = \tau$  (note  $t$  is just  $X^0$ ). Then the gauge condition (2.4) becomes

$$\dot{\mathbf{X}}^2 + \mathbf{X}'^2 = 1, \quad \dot{\mathbf{X}} \cdot \mathbf{X}' = 0 \quad (2.6)$$

while equation (2.5) now reads

$$\ddot{\mathbf{X}} - \mathbf{X}'' = 0 \quad (2.7)$$

where we have used  $\mathbf{X}$  to denote the  $(D - 1)$ -component spatial vector. As can be seen, the general solution

$$\mathbf{X} = \frac{1}{2} [\mathbf{a}(u) + \mathbf{b}(v)], \quad (2.8)$$

with  $u = \sigma - \tau, v = \sigma + \tau$ , satisfies the e.o.m. for arbitrary  $\mathbf{a}, \mathbf{b}$ . The gauge condition (2.6) imposes the constraint that

$$\left(\mathbf{a}^{(1)}\right)^2 = 1, \quad \left(\mathbf{b}^{(1)}\right)^2 = 1 \quad (2.9)$$

where the superscript (1) means first derivative with respect to the argument. We shall consider a closed string with  $\sigma$  going from 0 to  $L$ . Then the periodic boundary condition

$\int_0^L d\sigma \mathbf{X}' = 0$  (we do not compactify the target space; hence there is no winding) together with our choice to work in the string's center of mass frame  $\int_0^L d\sigma \dot{\mathbf{X}} = 0$  implies that

$$\int_0^L d\sigma \mathbf{a}^{(1)} = 0, \quad \int_0^L d\sigma \mathbf{b}^{(1)} = 0 \quad (2.10)$$

Pictorially the constraints (2.9) and (2.10) mean that  $\mathbf{a}^{(1)}(u)$  and  $\mathbf{b}^{(1)}(v)$  are two curves on the sphere  $\mathbf{S}^{D-2}$  with their centers at the origin. The constraint (2.10) must hold at any time  $\tau$ , which implies that  $\mathbf{a}^{(1)}, \mathbf{b}^{(1)}$  (and hence  $\mathbf{a}, \mathbf{b}$ ) are periodic in their arguments with period  $L$ . Using the periodicity of  $\mathbf{a}, \mathbf{b}$ , we can show that

$$\mathbf{X}\left(\sigma + \frac{L}{2}, \tau + \frac{L}{2}\right) = \mathbf{X}(\sigma, \tau) \quad (2.11)$$

i.e., the string looks the same at  $\tau$  and  $\tau + \frac{L}{2}$ . Next let us give some interesting examples of solutions.

**The spiky solution.** One can take the following  $\mathbf{a}^{(1)}$  and  $\mathbf{b}^{(1)}$  [38]

$$\begin{aligned} \mathbf{a}^{(1)}(u) &= \cos\left(2\pi M \frac{u}{L}\right) \mathbf{e}_1 + \sin\left(2\pi M \frac{u}{L}\right) \mathbf{e}_2 \\ \mathbf{b}^{(1)}(v) &= -\cos\left(2\pi N \frac{v}{L}\right) \mathbf{e}_1 + \sin\left(2\pi N \frac{v}{L}\right) \mathbf{e}_2 \end{aligned} \quad (2.12)$$

Pictorially  $\mathbf{a}^{(1)}$  and  $-\mathbf{b}^{(1)}$  rotate with different angular frequencies. Whenever they cross each other,  $\mathbf{X}'$  vanishes,  $\dot{\mathbf{X}}$  has magnitude of unity, and the point on the string where this happens moves at the speed of light. This is how a spike is developed. When  $\frac{M}{N}$  is not an integer, the string will be self-intersecting. When  $\frac{N}{M}$  is an integer, more specifically, let us take  $M = 1, N = n - 1$ , the string is not self-intersecting, and  $n$  is the number of spikes. The solution given in eq. (2.12) has its spikes pointing outward. If we flip the sign of the first term in  $\mathbf{b}^{(1)}(v)$ , the spikes will be pointing inward. The spiky strings considered in [2, 3] are generalizations of these solutions to  $AdS_5 \times S^5$ .<sup>1</sup>

**The kinky solution.** Note that  $\mathbf{a}^{(1)}$  and  $\mathbf{b}^{(1)}$  do not have to be continuous functions of  $u$  and  $v$ . When they have discontinuities, kinks are developed on the string. Let us take [39]

$$\begin{aligned} \mathbf{a}^{(1)}(u) &= \mathbf{e}_1 \quad \text{when } mL \leq u \leq \left(m + \frac{1}{2}\right)L \\ &= -\mathbf{e}_1 \quad \text{when } \left(m + \frac{1}{2}\right)L \leq u \leq (m + 1)L \end{aligned} \quad (2.13)$$

with  $m \in \mathbf{Z}$ , and

$$\mathbf{b}^{(1)}(v) = \mathbf{e}_2 \quad \text{when } mL \leq v \leq \left(m + \frac{1}{2}\right)L$$

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<sup>1</sup>Just like their flat space cousins, the spiky strings in  $AdS_5 \times S^5$  also have continuous first derivatives w.r.t. to worldsheet coordinates. This can be easily seen by considering the special case of two-spikes, which is the familiar spinning folded string.

$$-\mathbf{e}_2 \quad \text{when} \quad \left(m + \frac{1}{2}\right)L \leq v \leq (m+1)L \quad (2.14)$$

with  $m \in \mathbf{Z}$ . It is easy to see that this choice of  $\mathbf{a}^{(1)}$  and  $\mathbf{b}^{(1)}$  satisfies the constraints (2.9), (2.10). The resulting solution  $\mathbf{X}$  lies in the  $X^1, X^2$  plane and follows periodically the sequence: square  $\rightarrow$  doubled line  $\rightarrow$  square  $\rightarrow \dots$ . In the remainder of this paper, we shall generalize this solution to strings on  $R_t \times S^5$ .

### 3. String solutions on $R_t \times S^5$

#### 3.1 Construction of the solution

We follow the notation of [6] in describing  $S^5$  as a hypersurface embedded in  $\mathbf{R}^6$ . The equations of motion and the constraints are

$$\begin{aligned} X_A X_A &= X_1^2 + \dots + X_6^2 = 1 \\ \partial_+ \partial_- X_A + (\partial_+ X_B \partial_- X_B) X_A &= 0 \\ \partial_+ X_A \partial_+ X_A &= \kappa^2, \quad \partial_- X_A \partial_- X_A = \kappa^2 \end{aligned} \quad (3.1)$$

where we have taken  $t = \kappa\tau$ , and defined  $\xi^\pm = \frac{1}{2}(\tau \pm \sigma)$  (and hence  $\partial_\pm = \partial_\tau \pm \partial_\sigma$ ). We define the new fields  $Z$ ,  $X$ , and  $Y$  which are parametrized by

$$\begin{aligned} Z &= X_1 + iX_2 = \cos \gamma e^{i\phi_1}, \\ X &= X_3 + iX_4 = \sin \gamma \cos \psi e^{i\phi_2}, \\ Y &= X_5 + iX_6 = \sin \gamma \sin \psi e^{i\phi_3} \end{aligned} \quad (3.2)$$

which corresponds to the metric

$$ds^2 = d\gamma^2 + \cos^2 \gamma d\phi_1^2 + \sin^2 \gamma (d\psi^2 + \cos^2 \psi d\phi_2^2 + \sin^2 \psi d\phi_3^2) \quad (3.3)$$

on  $S^5$ .

We now restrict our attention to the subsector of  $\psi = 0, \phi_1 = \phi_2 = \phi$ , which corresponds to

$$\begin{aligned} X_1 &= \cos \gamma \cos \phi, & X_2 &= \cos \gamma \sin \phi, & X_3 &= \sin \gamma \cos \phi, \\ X_4 &= \sin \gamma \sin \phi, & X_5 &= 0, & X_6 &= 0 \end{aligned} \quad (3.4)$$

(hence we are looking at an  $S^3 \subset S^5$ ) then the metric becomes

$$ds^2 = d\gamma^2 + d\phi^2 \quad (3.5)$$

which is simply a flat metric on a torus.

Denoting  $\partial_\pm \gamma$  as  $\gamma_\pm$  and so on, one finds that the constraints become

$$\partial_+ X_A \partial_+ X_A = \gamma_+^2 + \phi_+^2 = \kappa^2, \quad \partial_- X_A \partial_- X_A = \gamma_-^2 + \phi_-^2 = \kappa^2 \quad (3.6)$$

and using

$$\partial_+ X_B \partial_- X_B = \gamma_+ \gamma_- + \phi_+ \phi_- \quad (3.7)$$

one finds the equations of motion become

$$\begin{aligned} \partial_+ \partial_- X_1 + (\partial_+ X_B \partial_- X_B) X_1 &= (\gamma_+ \phi_- + \gamma_- \phi_+) X_4 - \gamma_{+-} X_3 - \phi_{+-} X_2 = 0 \\ \partial_+ \partial_- X_2 + (\partial_+ X_B \partial_- X_B) X_2 &= -(\gamma_+ \phi_- + \gamma_- \phi_+) X_3 - \gamma_{+-} X_4 + \phi_{+-} X_1 = 0 \\ \partial_+ \partial_- X_3 + (\partial_+ X_B \partial_- X_B) X_3 &= -(\gamma_+ \phi_- + \gamma_- \phi_+) X_2 + \gamma_{+-} X_1 - \phi_{+-} X_4 = 0 \\ \partial_+ \partial_- X_4 + (\partial_+ X_B \partial_- X_B) X_4 &= (\gamma_+ \phi_- + \gamma_- \phi_+) X_1 + \gamma_{+-} X_2 + \phi_{+-} X_3 = 0 \end{aligned} \quad (3.8)$$

which are satisfied if we take

$$\gamma_{+-} = 0, \quad \phi_{+-} = 0, \quad \gamma_+ \phi_- + \gamma_- \phi_+ = 0 \quad (3.9)$$

(We note that (3.9) is the sufficient condition for (3.8), but probably not the necessary condition— naively one would expect to get just two equations for  $\gamma$  and  $\phi$ , not three — which means the solutions we shall present below are probably not exhaustive. <sup>2</sup> )

Setting

$$\gamma = \Gamma(\xi^+) + \tilde{\Gamma}(\xi^-), \quad \phi = \Phi(\xi^+) + \tilde{\Phi}(\xi^-) \quad (3.10)$$

with  $\Gamma, \tilde{\Gamma}, \Phi, \tilde{\Phi}$  being arbitrary functions satisfies  $\gamma_{+-} = 0$  and  $\phi_{+-} = 0$ . The equation  $\gamma_+ \phi_- + \gamma_- \phi_+ = 0$  is now satisfied if

$$\Gamma'(\xi^+) \tilde{\Phi}'(\xi^-) + \tilde{\Gamma}'(\xi^-) \Phi'(\xi^+) = 0 \quad (3.11)$$

(where primes stand for derivatives) which implies that

$$\Phi'(\xi^+) = q \Gamma'(\xi^+), \quad \tilde{\Phi}'(\xi^-) = -q \tilde{\Gamma}'(\xi^-) \quad (3.12)$$

with  $q$  being an arbitrary constant and the conformal constraints become

$$[\Gamma'(\xi^+)]^2 = \frac{\kappa^2}{1+q^2}, \quad [\tilde{\Gamma}'(\xi^-)]^2 = \frac{\kappa^2}{1+q^2} \quad (3.13)$$

We shall take the range of  $\sigma$  to be  $[0, L]$ . Since we shall only consider solutions without winding, we have the periodic boundary condition  $\gamma(\tau, \sigma) = \gamma(\tau, \sigma + L)$  and  $\phi(\tau, \sigma) = \phi(\tau, \sigma + L)$ , which are satisfied if we take  $\Gamma, \tilde{\Gamma}, \Phi, \tilde{\Phi}$  to be periodic in their arguments with the period being  $\frac{L}{2}$ . In summary, the solutions are given by (3.10) with the conditions (3.12) and (3.13), as well as the periodic boundary condition just mentioned. It is worth pointing out that, due to the periodicity of  $\Gamma, \tilde{\Gamma}, \Phi$  and  $\tilde{\Phi}$ ,  $\Gamma', \tilde{\Gamma}', \Phi'$  and  $\tilde{\Phi}'$  must integrate to zero over one period. This means that  $\Gamma'$  and  $\tilde{\Gamma}'$  must jump between  $\frac{\kappa}{\sqrt{1+q^2}}$  and  $-\frac{\kappa}{\sqrt{1+q^2}}$ , while  $\Phi'$  and  $\tilde{\Phi}'$  must jump between  $\frac{q\kappa}{\sqrt{1+q^2}}$  and  $-\frac{q\kappa}{\sqrt{1+q^2}}$ , i.e. they are discontinuous. As we shall

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<sup>2</sup>In particular, by insisting on (3.9), we might be excluding the generalizations of the flat space pulsating polygonal solutions which shall be discussed in section 7.

see, this discontinuity exhibited by the first derivatives of the embedding map  $X$  is a most salient feature of our solutions.

It is easy to show that for the above solution,  $\gamma(\tau + \frac{L}{2}, \sigma + \frac{L}{2}) = \gamma(\tau, \sigma)$  and  $\phi(\tau + \frac{L}{2}, \sigma + \frac{L}{2}) = \phi(\tau, \sigma)$ , i.e. the string looks the same at  $\tau$  and  $\tau + \frac{L}{2}$ . Besides, two limits we shall often consider are the  $q \rightarrow 0$  limit, in which the string becomes folded along the  $\gamma$ -axis (i.e.  $\phi = 0$ ), and the  $q \rightarrow \infty$  limit, in which the string becomes folded along the  $\phi$ -axis (i.e.  $\gamma = 0$ ).

Let us compute the charges of the solution. The energy is given by

$$E = S_{50} = \sqrt{\lambda} \int_0^L \frac{d\sigma}{L} (Y_5 \dot{Y}_0 - Y_0 \dot{Y}_5) \quad (3.14)$$

with  $Y_5 = \cos t$  and  $Y_0 = \sin t$  being two of the  $AdS_5$  embedding coordinates (the other four  $Y_{1,2,3,4}$  all vanish), and one finds  $E = \sqrt{\lambda} \kappa$ . The angular momenta on the  $S^5$  are given by

$$J_{AB} = \sqrt{\lambda} \int_0^L \frac{d\sigma}{L} (X_A \dot{X}_B - X_B \dot{X}_A) \quad (3.15)$$

One finds

$$\begin{aligned} X_1 \dot{X}_2 - X_2 \dot{X}_1 &= \frac{1}{2}(\phi_+ + \phi_-) \cos^2 \gamma = q\gamma' \cos^2 \gamma \\ X_1 \dot{X}_3 - X_3 \dot{X}_1 &= \frac{1}{2}(\gamma_+ + \gamma_-) \cos^2 \phi = \frac{1}{q}\phi' \cos^2 \phi \\ X_1 \dot{X}_4 - X_4 \dot{X}_1 &= \frac{1}{2} \sin \phi \cos \phi (\gamma_+ + \gamma_-) + \frac{1}{2} \sin \gamma \cos \gamma (\phi_+ + \phi_-) \\ &= \frac{1}{q}\phi' \sin \phi \cos \phi + q\gamma' \sin \gamma \cos \gamma \\ X_2 \dot{X}_3 - X_3 \dot{X}_2 &= \frac{1}{2} \sin \phi \cos \phi (\gamma_+ + \gamma_-) - \frac{1}{2} \sin \gamma \cos \gamma (\phi_+ + \phi_-) \\ &= \frac{1}{q}\phi' \sin \phi \cos \phi - q\gamma' \sin \gamma \cos \gamma \\ X_2 \dot{X}_4 - X_4 \dot{X}_2 &= \frac{1}{2} \sin^2 \phi (\gamma_+ + \gamma_-) = \frac{1}{q}\phi' \sin^2 \phi \\ X_3 \dot{X}_4 - X_4 \dot{X}_3 &= \frac{1}{2} \sin^2 \gamma (\phi_+ + \phi_-) = q\gamma' \sin^2 \gamma \end{aligned} \quad (3.16)$$

where in the last steps of the above equations we have made use of (3.10) and (3.12) and hence  $\dot{\gamma} = \frac{1}{2}(\gamma_+ + \gamma_-) = \frac{1}{q}\phi'$  and  $\dot{\phi} = \frac{1}{2}(\phi_+ + \phi_-) = q\gamma'$ . This means, for finite nonzero  $q$ , all of the above expressions are total derivatives of functions of either  $\gamma$  or  $\phi$  w.r.t to  $\sigma$  and thus  $J_{AB} = 0$ . For  $q = 0$  or  $q \rightarrow \infty$  one can also show that  $J_{AB} = 0$ , using the fact that  $\int_0^L d\sigma \gamma_+ = \int_0^L d\sigma \Gamma'(\xi^+) = 2 \int_{\tau/2}^{\tau/2+L/2} d\xi^+ \Gamma'(\xi^+) = 0$  (see the paragraph below equation (3.13)), and similarly  $\int_0^L d\sigma \gamma_- = 0$ ,  $\int_0^L d\sigma \phi_+ = 0$ , and  $\int_0^L d\sigma \phi_- = 0$ . In summary, the string solutions we consider do not carry any angular momentum.

### 3.2 Higher charges of the solutions via Bäcklund transformation

In this subsection we investigate the infinite number of local commuting charges generated via Bäcklund transformation, following [40] and [23]. To avoid confusion with the spher-



ical coordinate, we shall use  $s$ , instead of the commonly used  $\gamma$ , to denote the spectral parameter. The Bäcklund transformation is given by

$$\begin{aligned} 2s^2\partial_+(X(s) + X) &= (1 + s^2)(X(s) \cdot \partial_+X)(X(s) - X) \\ 2\partial_-(X(s) - X) &= -(1 + s^2)(X(s) \cdot \partial_-X)(X(s) + X) \end{aligned} \quad (3.17)$$

together with the normalization conditions

$$X(s=0) = X, \quad X(s) \cdot X(s) = 1, \quad X(s) \cdot X = \frac{1 - s^2}{1 + s^2} \quad (3.18)$$

It is readily shown that  $X(s)$  satisfies the same e.o.m.'s and constraints (3.1) as  $X$ .

Local conservation laws are derived by contracting the first equation of (3.17) with  $\partial_-X$  and the second equation with  $\partial_+X$  and then adding the two equations, giving

$$\partial_\tau (s^2X(s) \cdot \partial_-X + X(s) \cdot \partial_+X) + \partial_\sigma (s^2X(s) \cdot \partial_-X - X(s) \cdot \partial_+X) = 0 \quad (3.19)$$

which leads to the conserved quantity

$$\mathcal{E}(s) \equiv \int_0^L \frac{d\sigma}{2L} [sX(s) \cdot \partial_+X + s^3X(s) \cdot \partial_-X] \quad (3.20)$$

Expanding  $X(s)$  as a power series in  $s$

$$X(s) = \sum_{k=0}^{\infty} X^{(k)} s^k \quad (3.21)$$

we can use  $\mathcal{E}(s)$  as a generating function of an infinite number of local charges  $\mathcal{E}_k$

$$\mathcal{E}(s) = \sum_{k=2}^{\infty} \mathcal{E}_k s^k \quad (3.22)$$

where

$$\mathcal{E}_k = \int_0^L \frac{d\sigma}{2L} [X^{(k-1)} \cdot \partial_+X + X^{(k-3)} \cdot \partial_-X], \quad k \geq 2 \quad (3.23)$$

and  $X^{(k)} = 0$  for  $k$  negative.

To evaluate the local charges it is preferable to have an exact solution for  $X(s)$ , when this is unavailable can find the solution as a power series expansion in  $s$ . Equations (3.16) to (3.18) of [23] give such an expression for a perturbative solution  $X(s) = \sum X^{(k)} s^k$  using as their trial or bare solutions,  $X$ , the folded or circular string. However, when the  $X$ 's are themselves continuous but have discontinuous  $\partial_+X$  and  $\partial_-X$  (which is the case for the kinky solutions that are of primary interest in this paper), that expression is incorrect and gives discontinuous and even singular  $X^{(k)}$  for higher  $k$ 's. We shall now illustrate this with the  $q = 0$  case, which gives  $\phi = 0$ , and where we will be able to find an exact solution to the Bäcklund transformation. Our solution corresponds to

$$X_1 = \cos \gamma, \quad X_3 = \sin \gamma, \quad X_2 = X_4 = X_5 = X_6 = 0 \quad (3.24)$$

and is simply the  $O(2)$  sigma model. Now the normalization condition alone determines the solution. Let us make the ansatz

$$X_1(s) = \cos \gamma(s), \quad X_3(s) = \sin \gamma(s), \quad X_2(s) = X_4(s) = X_5(s) = X_6(s) = 0 \quad (3.25)$$

then

$$X(s) \cdot X = \cos(\gamma(s) - \gamma) = \frac{1 - s^2}{1 + s^2} \quad (3.26)$$

Defining  $s = \tan \frac{\beta}{2}$ , we obtain an <sup>3</sup> exact solution to the Backlund transformation

$$\gamma(s) = \gamma + \beta = \gamma + 2 \arctan(s) \quad (3.27)$$

giving

$$X_1(s) = \frac{1 - s^2}{1 + s^2} \cos \gamma - \frac{2s}{1 + s^2} \sin \gamma, \quad X_3(s) = \frac{1 - s^2}{1 + s^2} \sin \gamma + \frac{2s}{1 + s^2} \cos \gamma \quad (3.28)$$

It can be explicitly checked that the above  $X(s)$  satisfies (3.17). It is clear that  $X(s)$  corresponds to a rotation of  $X$  in the 13 plane. Hence we see that  $X(s)$  indeed admits a power expansion around  $s = 0$ :  $X(s) = \sum_{k=0}^{\infty} X^{(k)} s^k$  with  $X^{(k)} = \frac{1}{k!} \frac{d^k X(s)}{ds^k} |_{s=0}$ , which gives

$$X_1^{(1)} = -2 \sin \gamma, \quad X_1^{(2)} = -2 \cos \gamma, \quad X_1^{(3)} = 2 \sin \gamma, \quad \dots \quad (3.29)$$

Let us compare this with the perturbative solution given by (3.16) to (3.18) of [23], which we denote as  $\tilde{X}^{(k)}$

$$\begin{aligned} \tilde{X}_1^{(1)} &= \frac{2\partial_+ X_1}{\kappa} = -2 \sin \gamma \frac{\partial_+ \gamma}{\kappa} \\ \tilde{X}_1^{(2)} &= \frac{2\partial_+^2 X_1}{\kappa^2} = -2 \cos \gamma \frac{(\partial_+ \gamma)^2}{\kappa^2} - 2 \sin \gamma \frac{\partial_+^2 \gamma}{\kappa^2} = -2 \cos \gamma - 2 \sin \gamma \frac{\partial_+^2 \gamma}{\kappa^2} \end{aligned} \quad (3.30)$$

where in the second equation we have used the fact  $(\partial_+ \gamma)^2 = \kappa^2$  (see (3.13)). We see that  $\tilde{X}_1^{(1)}$  differs from  $X_1^{(1)}$  by the factor  $\frac{\partial_+ \gamma}{\kappa}$  (which alternates between  $+1$  and  $-1$  over one period) and is thus discontinuous,  $\tilde{X}_1^{(2)}$  differs from  $X_1^{(2)}$  by a term involving  $\partial_+^2 \gamma$  and thus exhibits delta-functional singularity. We omit the explicit expressions here, but one can readily check that  $\tilde{X}_1^{(3)}$  differs from  $X_1^{(3)}$  by the factor  $\frac{\partial_+ \gamma}{\kappa}$  as well as terms involving  $\partial_+^2 \gamma$  and  $\partial_+^3 \gamma$  and is thus more singular. Similarly, higher and higher  $\tilde{X}_1^{(k)}$ 's contain higher derivatives of  $\gamma$  and exhibit worse singular behavior. Hence we see that  $\tilde{X}_1^{(k)}$  cannot be the right solution because the string should have a continuous profile order by order in  $s$ . This illustrates a subtlety when solving the Bäcklund transformation order by order in  $s$ : at each order there might be more than one solution satisfying equation (3.17) (both

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<sup>3</sup>There is at least one other solution which can be obtained by letting  $s \rightarrow -s$ . It is clear that this is a solution as the Bäcklund equations are even in  $s$ . In our case it is straightforward to also find this solution and the charges are trivially related to those described below.

$X_1^{(k)}$  and  $\tilde{X}_1^{(k)}$  above satisfy the equation), and one has to choose the solution that gives a continuous string profile at each order (in our case it is  $X_1^{(k)}$ ).

Let us now compute the generating function of local charges  $\mathcal{E}(s)$  for the  $q = 0$  case. One readily finds, using the exact solution  $X(s)$  we have

$$\mathcal{E}(s) = \int_0^L \frac{d\sigma}{2L} [sX(s) \cdot \partial_+ X + s^3 X(s) \cdot \partial_- X] = \int_0^L \frac{d\sigma}{2L} (s\partial_+ \gamma + s^3 \partial_- \gamma) \sin \beta = 0 \quad (3.31)$$

where in the last step we have used the fact that

$$\int_0^L d\sigma \partial_+ \gamma = \int_0^L d\sigma \Gamma'(\xi^+) = 2 \int_{\tau/2}^{\tau/2+L/2} d\xi^+ \Gamma'(\xi^+) = 0$$

(see the paragraph under (3.13)) and similarly  $\int_0^L d\sigma \partial_- \gamma = 0$ . Hence we see that all the higher charges  $\mathcal{E}_k$  vanish. The  $q = \infty$  case is completely analogous.

The above construction can be generalized to the case of nonzero and finite  $q$ . We make the ansatz

$$\begin{aligned} X_1(s) &= \cos \gamma(s) \cos \phi(s), & X_2(s) &= \cos \gamma(s) \sin \phi(s), & X_3(s) &= \sin \gamma(s) \cos \phi(s), \\ X_4(s) &= \sin \gamma(s) \sin \phi(s), & X_5(s) &= X_6(s) = 0 \end{aligned} \quad (3.32)$$

and from the normalization condition we have,

$$\begin{aligned} X(s) \cdot X &= \cos(\gamma(s) - \gamma) \cos(\phi(s) - \phi) \\ &= \frac{1 - s^2}{1 + s^2}. \end{aligned} \quad (3.33)$$

We further assume that  $\gamma(s) = \gamma + a(s)$  and  $\phi(s) = \phi + b(s)$ , where  $a, b$  are independent of  $(\tau, \sigma)$ . Then the normalization condition becomes

$$\cos a \cos b = \frac{1 - s^2}{1 + s^2} \quad (3.34)$$

Also notice

$$\begin{aligned} X(s) \cdot \partial_+ X &= \gamma_+ \sin a \cos b + \phi_+ \cos a \sin b \\ X(s) \cdot \partial_- X &= \gamma_- \sin a \cos b + \phi_- \cos a \sin b \end{aligned} \quad (3.35)$$

Using the facts  $s^2 = \frac{1 - \cos a \cos b}{1 + \cos a \cos b}$ ,  $\phi_+ = q\gamma_+$ ,  $\phi_- = -q\gamma_-$ , and going through a series of trigonometric identities, we can simplify the Bäcklund transformation (3.17) and find that it is satisfied when

$$q \sin a = \sin b \quad (3.36)$$

For example, the first equation in (3.17) for  $A = 1$  becomes

$$[\sin \gamma \sin \phi(s) - \sin \gamma(s) \sin \phi](q \sin a - \sin b) = 0 \quad (3.37)$$

Then using equation (3.35) and the fact  $\int_0^L d\sigma \gamma_{\pm} = 0$ ,  $\int_0^L d\sigma \phi_{\pm} = 0$ , we conclude that  $\mathcal{E}(s)$  defined in (3.20) vanishes identically.

In summary, we have found the exact solution to the Bäcklund transformation, which corresponds to shifting  $\gamma$  and  $\phi$  by  $a(s)$  and  $b(s)$  respectively, with  $a, b$  determined by  $\cos a \cos b = \frac{1 - s^2}{1 + s^2}$  and  $q \sin a = \sin b$ . The generating function  $\mathcal{E}(s)$  turns out to always vanish, hence so do all the local charges  $\mathcal{E}_k$ .

### 3.3 Higher charges via monodromy matrix

A second method for constructing conserved charges of an integrable systems is to expand the eigenvalues of the monodromy matrix in powers of the spectral parameter about its singular points. We shall work in the formulation of SU(2) principal chiral model following [17]. The SU(2) principal chiral field is

$$g = \begin{pmatrix} X_1 + iX_2 & X_3 + iX_4 \\ -X_3 + iX_4 & X_1 - iX_2 \end{pmatrix} = \begin{pmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{pmatrix} \in \text{SU}(2) \quad (3.38)$$

The  $su(2)$ -algebra valued right currents are defined as  $j = -g^{-1}dg$  (notice that our definition of  $j$  differs from that of [17] by an overall sign), which can be shown to satisfy

$$\partial_+ j_- + \partial_- j_+ = 0, \quad \partial_+ j_- - \partial_- j_+ - [j_+, j_-] = 0 \quad (3.39)$$

One can define a one-parameter family of currents  $J(x)$

$$J_{\pm}(x) = \frac{j_{\pm}}{1 \mp x} \quad (3.40)$$

which has vanishing curvature

$$\partial_+ J_- - \partial_- J_+ - [J_+, J_-] = 0 \quad (3.41)$$

due to eq.'s (3.39). Let us consider the Wilson line defined as

$$\Omega(\sigma_1, \sigma_2) \equiv P e^{\int_{\sigma_2}^{\sigma_1} d\sigma J_{\sigma}(\sigma)} = P e^{\int_{\sigma_2}^{\sigma_1} d\sigma \frac{1}{2} \left( \frac{j_+}{1-x} - \frac{j_-}{1+x} \right)} \quad (3.42)$$

where  $\sigma_1 > \sigma_2$  and  $P$  stands for path ordering, putting larger values of  $\sigma$  on the left.  $\Omega(\sigma_1, \sigma_2)$  as defined above has the properties

$$\partial_{\sigma_1} \Omega(\sigma_1, \sigma_2) = J_{\sigma}(\sigma_1) \Omega(\sigma_1, \sigma_2) \quad (3.43)$$

and

$$\Omega(\sigma_1, \sigma_2) \Omega(\sigma_2, \sigma_3) = \Omega(\sigma_1, \sigma_3), \quad \sigma_1 > \sigma_2 > \sigma_3 \quad (3.44)$$

The monodromy matrix is the Wilson loop  $\Omega(L, 0)$ . The vanishing of the curvature of  $J$  implies that under time evolution the monodromy matrix only undergoes conjugation, hence its eigenvalues are conserved quantities.

For the solutions constructed in section 3.1,

$$g = \begin{pmatrix} \cos \gamma e^{i\phi} & \sin \gamma e^{i\phi} \\ -\sin \gamma e^{-i\phi} & \cos \gamma e^{-i\phi} \end{pmatrix} \quad (3.45)$$

and

$$j = -i [\sigma^1 (\sin 2\gamma) d\phi + \sigma^2 d\gamma + \sigma^3 (\cos 2\gamma) d\phi] \quad (3.46)$$

with  $\sigma^{1,2,3}$  being the standard Pauli matrices. Let us first consider the simplest case of  $q = 0$  and  $q = \infty$ . In the  $q = 0$  case,  $\gamma \neq 0$  and  $\phi = 0$ . This leads to  $j = -i\sigma^2 d\gamma$ , and  $J_\sigma = \frac{1}{2} \left( \frac{\gamma_+}{1-x} - \frac{\gamma_-}{1+x} \right) (-i\sigma^2)$ . Since  $J_\sigma$ 's at different values of  $\sigma$  commute, no path ordering is necessary. This gives

$$\Omega(\sigma_1, \sigma_2) = e^{\int_{\sigma_2}^{\sigma_1} d\sigma \frac{1}{2} \left( \frac{\gamma_+}{1-x} - \frac{\gamma_-}{1+x} \right) (-i\sigma^2)} \quad (3.47)$$

Due to the fact  $\int_0^L d\sigma \gamma_\pm = 0$ , the monodromy matrix  $\Omega(L, 0) = \mathbf{1}_{2 \times 2}$  in this case. The  $q = \infty$  case is similar. Now  $j = -i\sigma^3 d\phi$ ,  $J_\sigma = \frac{1}{2} \left( \frac{\phi_+}{1-x} - \frac{\phi_-}{1+x} \right) (-i\sigma^3)$ , and

$$\Omega(\sigma_1, \sigma_2) = e^{\int_{\sigma_2}^{\sigma_1} d\sigma \frac{1}{2} \left( \frac{\phi_+}{1-x} - \frac{\phi_-}{1+x} \right) (-i\sigma^3)} \quad (3.48)$$

which again leads to  $\Omega(L, 0) = \mathbf{1}_{2 \times 2}$ , using the fact  $\int_0^L d\sigma \phi_\pm = 0$ .

Next let us consider a general nonzero finite value of  $q$ . Using the facts  $\phi_+ = q\gamma_+$ ,  $\phi_- = -q\gamma_-$ , we find

$$j_\pm = \mp i\gamma_\pm (\sigma^1 q \sin 2\gamma \pm \sigma^2 + \sigma^3 q \cos 2\gamma) \quad (3.49)$$

and hence

$$J_\sigma = -i (\sigma^1 q f \sin 2\gamma + \sigma^2 w + \sigma^3 q f \cos 2\gamma) \quad (3.50)$$

where we have defined the functions  $f, w$  as

$$f = \frac{1}{2} \left( \frac{\gamma_+}{1-x} + \frac{\gamma_-}{1+x} \right), \quad w = \frac{1}{2} \left( \frac{\gamma_+}{1-x} - \frac{\gamma_-}{1+x} \right) \quad (3.51)$$

One salient feature of our solutions is the discontinuities exhibited in  $\gamma_\pm$  (see the paragraph below equation (3.13)). Thus we divide the string into segments. In each segment,  $\gamma_\pm$  are constant, and so  $\partial_\sigma \gamma = \frac{1}{2}(\gamma_+ - \gamma_-)$  is also constant. This means that in each segment  $f, w$  are constant and the  $\sigma$ -dependence of  $J_\sigma$  comes purely from  $\gamma$ , which is a linear function of  $\sigma$ . Going from one segment to the next,  $\gamma_+$ ,  $\gamma_-$ , or both, jump and consequently  $f, w$  and  $\partial_\sigma \gamma$  jump from one value to another. This suggests we should compute the monodromy matrix in a piecewise manner

$$\Omega(L, 0) = \Omega(\sigma_0 = L, \sigma_1) \Omega(\sigma_1, \sigma_2) \dots \Omega(\sigma_{m-1}, \sigma_m = 0) \quad (3.52)$$

where  $\sigma_1, \sigma_2, \dots, \sigma_{m-1}$  are values of  $\sigma$  at which  $\gamma_+$ ,  $\gamma_-$ , or both, jump. Let us solve

$$\partial_\sigma \Omega(\sigma, \sigma_{k+1}) = J_\sigma(\sigma) \Omega(\sigma, \sigma_{k+1}), \quad \sigma_{k+1} < \sigma < \sigma_k \quad (3.53)$$

for  $k = 0, 1, \dots, m-1$ . This can be done by defining the auxiliary 2-component ‘‘wave-function’’  $\Psi(\sigma)$

$$\Psi(\sigma) \equiv \Omega(\sigma, \sigma_{k+1}) \Psi(\sigma_{k+1}) \quad (3.54)$$

with  $\Psi(\sigma_{k+1})$  being an arbitrary constant 2-component vector. We see that

$$\partial_\sigma \Psi(\sigma) = J_\sigma(\sigma) \Psi(\sigma) \quad (3.55)$$

Given the expression of  $J_\sigma$  in (3.50) and the fact that  $f, w, \partial_\sigma \gamma$  are all constant in the interval  $(\sigma_{k+1}, \sigma_k)$ , this problem is simply a textbook one of solving the Schrodinger equation for a spin-1/2 particle in a rotating magnetic field. Using the identity  $\sigma^3 \cos 2\gamma + \sigma^1 \sin 2\gamma = e^{-i\gamma\sigma^2} \sigma^3 e^{i\gamma\sigma^2}$  and going to the rotating frame  $\Psi = e^{-i\gamma\sigma^2} \tilde{\Psi}$ , one readily solves this Schrodinger equation and finds

$$\Omega(\sigma, \sigma_{k+1}) = e^{-i\gamma(\sigma)\sigma^2} e^{-i\{(w^{(k)} - \partial_\sigma \gamma^{(k)})\sigma^2 + qf^{(k)}\sigma^3\}(\sigma - \sigma_{k+1})} e^{i\gamma(\sigma_{k+1})\sigma^2} \quad (3.56)$$

where we have added the superscript  $(k)$  to  $f, w, \partial_\sigma \gamma$  to make it explicit that these are their (constant) values in the segment  $\sigma_{k+1} < \sigma < \sigma_k$ . Thus we have

$$\Omega(\sigma_k, \sigma_{k+1}) = e^{-i\gamma(\sigma_k)\sigma^2} e^{-i\{(w^{(k)} - \partial_\sigma \gamma^{(k)})\sigma^2 + qf^{(k)}\sigma^3\}(\sigma_k - \sigma_{k+1})} e^{i\gamma(\sigma_{k+1})\sigma^2} \quad (3.57)$$

and in turn

$$\begin{aligned} \Omega(L, 0) &= e^{-i\gamma(\sigma_0)\sigma^2} e^{-i\{(w^{(0)} - \partial_\sigma \gamma^{(0)})\sigma^2 + qf^{(0)}\sigma^3\}(\sigma_0 - \sigma_1)} \dots \\ &\quad e^{-i\{(w^{(m-1)} - \partial_\sigma \gamma^{(m-1)})\sigma^2 + qf^{(m-1)}\sigma^3\}(\sigma_{m-1} - \sigma_m)} e^{i\gamma(\sigma_m)\sigma^2} \end{aligned}$$

One final simplification comes from the observation that

$$w - \partial_\sigma \gamma = \frac{1}{2} \left( \frac{\gamma_+}{1-x} - \frac{\gamma_-}{1+x} \right) - \frac{1}{2}(\gamma_+ - \gamma_-) = x \frac{1}{2} \left( \frac{\gamma_+}{1-x} + \frac{\gamma_-}{1+x} \right) = xf \quad (3.58)$$

which implies

$$(w^{(k)} - \partial_\sigma \gamma^{(k)})\sigma^2 + qf^{(k)}\sigma^3 = f^{(k)}(x\sigma^2 + q\sigma^3), \quad (3.59)$$

which commute between different segments. Using this fact, we find

$$\Omega(L, 0) = e^{-i\gamma(\sigma_0)\sigma^2} e^{-i(x\sigma^2 + q\sigma^3) \sum_{k=0}^{m-1} f^{(k)}(\sigma_k - \sigma_{k+1})} e^{i\gamma(\sigma_m)\sigma^2} \quad (3.60)$$

and we note that

$$\sum_{k=0}^{m-1} f^{(k)}(\sigma_k - \sigma_{k+1}) = \int_0^L df = \int_0^L d\sigma \frac{1}{2} \left( \frac{\gamma_+}{1-x} + \frac{\gamma_-}{1+x} \right) = 0 \quad (3.61)$$

using  $\int_0^L d\sigma \gamma_\pm = 0$ . Also noticing that  $\gamma(\sigma_0 = L) = \gamma(\sigma_m = 0)$ , we finally conclude that

$$\Omega(L, 0) = \mathbf{1}_{2 \times 2} \quad (3.62)$$

In summary, we have shown that the monodromy matrix is simply the identity matrix for all the solutions constructed in section 3.1.

Now that we have shown for the kinky solutions the monodromy matrix is trivial, and hence the quasi-momentum  $p(x)$  defined by  $\text{tr} \Omega(L, 0; x) = 2 \cos p(x)$  is identically zero (modulo  $2\pi m$ ,  $m$  being an integer), we would like to comment on the previous constructions which found that the quasi-momentum has poles at  $x = \pm 1$  with the residue being  $-\pi\kappa$ .

The argument behind this claim is as follows (see e.g. [19]). One can diagonalize  $j_{\pm}$  to  $j_{\pm}^{\text{diag}}$ , then as  $x \rightarrow \pm 1$  we can drop the path ordering and

$$\begin{aligned} \text{tr}\Omega(L, 0; x) &\rightarrow \text{tr}\left(e^{\int_0^L d\sigma \frac{\pm 1}{2} \frac{j_{\pm}^{\text{diag}}}{1 \mp x}}\right) = \text{tr}\begin{pmatrix} \exp\left(\frac{\pm 1}{2(1 \mp x)} \int_0^L d\sigma i\lambda\right) & 0 \\ 0 & \exp\left(\frac{\pm 1}{2(1 \mp x)} \int_0^L d\sigma (-i\lambda)\right) \end{pmatrix} \\ &= 2 \cos\left(\frac{1}{2(1 \mp x)} \int_0^L d\sigma \lambda\right) \end{aligned} \tag{3.63}$$

where we have written

$$j_{\pm}^{\text{diag}} = \begin{pmatrix} i\lambda & 0 \\ 0 & -i\lambda \end{pmatrix} \tag{3.64}$$

This implies that  $p(x)$  has a pole at  $x = \pm 1$  with the residue being  $-\frac{1}{2} \int_0^L d\sigma \lambda$ . Meanwhile notice that

$$\text{tr}(j_{\pm}^2) = -2\partial_+ X_A \partial_+ X_A = -2\kappa^2 \tag{3.65}$$

by virtue of the conformal gauge constraint, which implies  $\lambda^2 = \kappa^2$ . The argument is then that  $\lambda$  is a  $\sigma$ -independent constant, either  $\kappa$  or  $-\kappa$ , over the entire string length and this then gives a residue  $-\frac{L\kappa}{2} = -\pi\kappa$  (setting  $L = 2\pi$ ). However, this construction excludes the case when  $j = -g^{-1}dg$  is not continuous (in fact it does not have to be continuous, since it involves first derivatives of string embedding  $X(\tau, \sigma)$ ). We can have step functions in  $j$  and still get a continuous string profile  $X$ , which seems to be a sensible solution. This means  $\lambda$  does not have to be a  $\sigma$ -independent constant: it can jump between  $\kappa$  and  $-\kappa$  as we move from one segment of the string to another. This is exactly what is happening for the kinky string. Take the  $q = \infty$  case for example (in this case,  $j_{\pm} = -i\sigma^3 \phi_{\pm}$  is already diagonal). We see that the eigenvalue  $\lambda = -\phi_{\pm}(\sigma)$ , which can have an arbitrary pattern of jumps and integrates to zero over the whole string length, giving a vanishing residue. The moral of the story for the  $q = 0$  and general nonzero finite  $q$  cases is the same.

Given that the monodromy matrix for our solutions is always trivial, it would seem that the usual local and non-local charges which are found by expanding in powers of the spectral parameter are all vanishing. It is interesting to see if we can find other, non-trivial, charges. The class of local charges constructed for principal chiral models based on the invariant tensors of the underlying algebra (see for example [41–43]) are one possibility. In particular, it is easy to see from the equations of motion (3.39) that

$$\partial_- \text{Tr}(j_+^m) = \partial_+ \text{Tr}(j_-^m) = 0 \tag{3.66}$$

for all  $m$ . These classically conserved charges are in involution with the non-local Yangian symmetries of the principal chiral model and indeed may survive to the quantum theory. Unfortunately for us while they are not all zero they are essentially trivial. This can be seen as follows: taking the diagonalized form of  $j_{\pm}$  as in (3.64), we see that for odd  $m$  the traces  $\text{Tr}(j_{\pm}^m)$  vanish, while for even  $m$  we get  $\text{Tr}(j_{\pm}^m) = 2(-1)^{m/2} \kappa^m$  (using the fact  $\lambda^2 = \kappa^2$ , see (3.65)), which are the same for all of our solutions. However, as mentioned

in [42] it is possible to consider differentials of these charges, for example it is clear from equation (3.39) that

$$Tr(j_+^m) \partial_+^p Tr(j_+^q) \quad \text{and} \quad Tr(j_-^m) \partial_-^p Tr(j_-^q) \tag{3.67}$$

are conserved. Of course as the traces are vanishing or constant we might suppose that these charges are also vanishing. However given the kinky feature of our solutions there is an ambiguity involved in their definition. Let us evaluate a simple case,  $Tr(j_+ \partial_+ j_+)$ , on our solution,

$$j_+ = -iq\gamma_+ \begin{pmatrix} \cos 2\gamma & \sin 2\gamma - \frac{i}{q} \\ \sin 2\gamma + \frac{i}{q} & -\cos 2\gamma \end{pmatrix}. \tag{3.68}$$

For our solutions  $\partial_+^2 \gamma = 0$  everywhere except at the kinks where

$$\partial_+^2 \gamma = \frac{\pm 2\kappa}{\sqrt{1+q^2}} \delta(\xi_{\text{kink}}^+) \tag{3.69}$$

with the sign corresponding to whether the value of  $\gamma_+$  increases or decreases as we cross the kink. We thus find the conserved charge

$$\begin{aligned} Q &= \int d\sigma Tr(j_+ \partial_+ j_+) \\ &= -4\kappa \sqrt{1+q^2} \sum_{\text{kinks}} \pm \gamma_+ |_{\text{kinks}}. \end{aligned} \tag{3.70}$$

This leads us to the question of what exactly is the value of  $\gamma_+$  at the kinks? In order for the Virasoro constraints to be satisfied it must be  $\frac{\pm \kappa}{\sqrt{1+q^2}}$  however the sign is somewhat arbitrary. If we choose it to be always positive we find the above charge to be vanishing for an even number of kinks. However if we choose the sign of  $\gamma_+$  at each kink to correspond to that of  $\partial_+^2 \gamma$  we get a non-vanishing result, which, while it is independent of  $q$  does depend on the number of kinks. Currents involving more derivatives will result in more singular charges while currents with more powers of  $j_{\pm}$  result in simple powers of the overall coefficient.

### 3.4 Example solutions

In this section we give some example solutions which have  $n$  evenly spaced jumps in the worldsheet first derivatives.

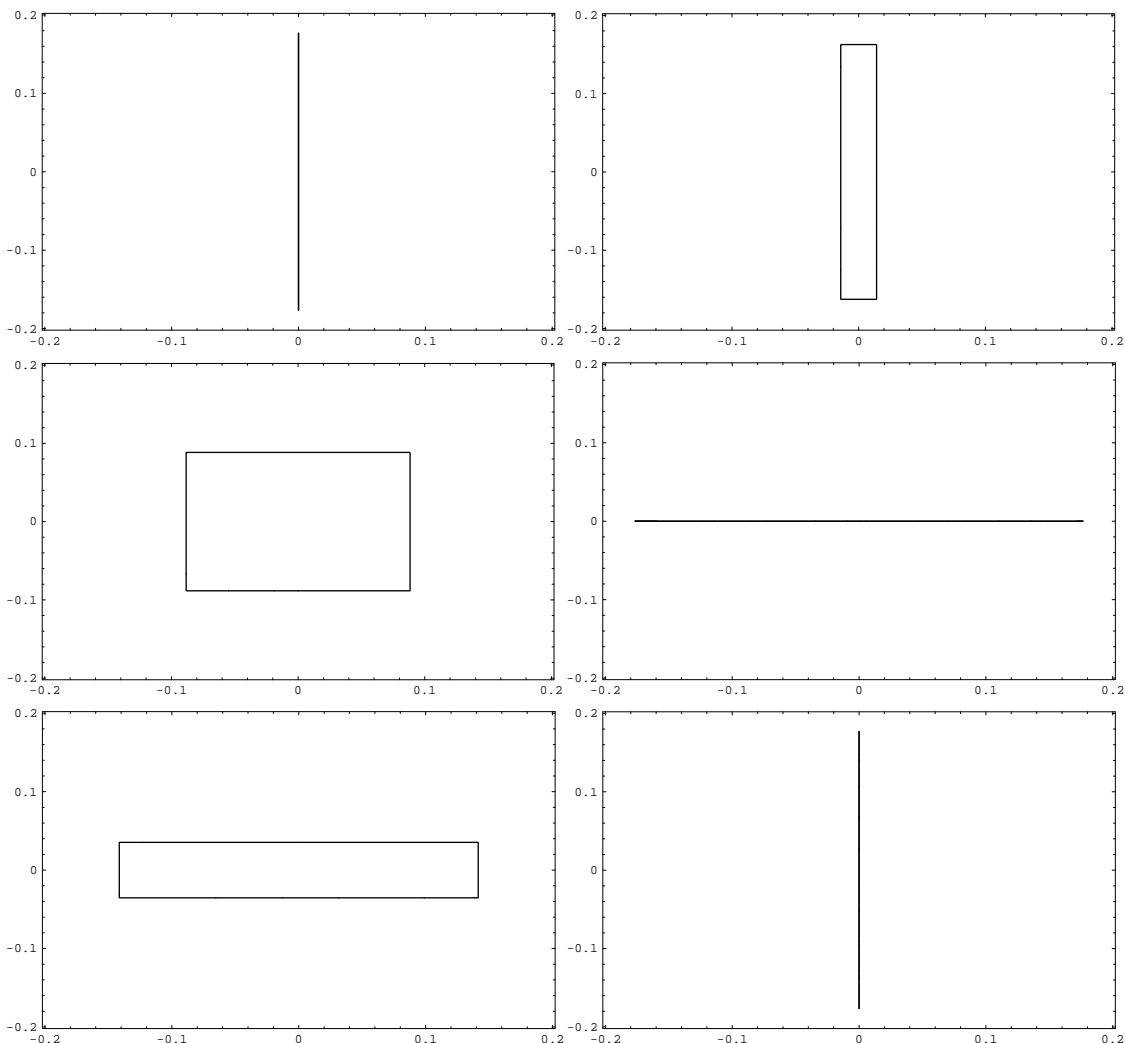
**Example I.** Take  $\Gamma'(\xi^+) = f(\xi^+)$  and  $\tilde{\Gamma}'(\xi^-) = f(\xi^-)$  with

$$\begin{aligned} f(\xi) &= \frac{\kappa}{\sqrt{1+q^2}} & \text{for } \frac{mL}{2} < \xi < \frac{mL}{2} + \frac{L}{4} \\ f(\xi) &= -\frac{\kappa}{\sqrt{1+q^2}} & \text{for } \frac{mL}{2} + \frac{L}{4} < \xi < \frac{mL}{2} + \frac{L}{2} \end{aligned} \tag{3.71}$$

( $m \in \mathbf{Z}$ ) This leads to

$$\gamma = F(\xi^+) + F(\xi^-), \quad \phi = q [F(\xi^+) - F(\xi^-)] \tag{3.72}$$

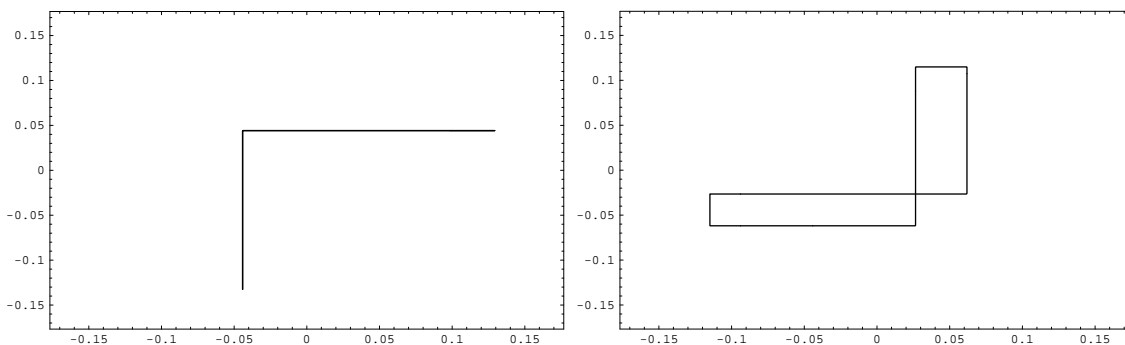




**Figure 1:** The pulsating rectangle solution for  $q = 1$ , at  $\tau = 0, 0.02L$  (first row, left to right),  $\tau = 0.125, 0.25$  (second row), and  $\tau = 0.3, 0.5$  (third row); horizontal axis is  $\phi$  and vertical axis is  $\gamma$ , both labeled in units of  $\kappa L$

with  $F(\xi) = \int_{L/8}^{\xi} f(\tilde{\xi}) d\tilde{\xi}$ . The time evolution of the string's profile is as follows: at  $\tau = 0$ , the string is folded and lies on the  $\gamma$ -axis, extending from  $\gamma = -\frac{\kappa}{\sqrt{1+q^2}} \frac{L}{4}$  to  $\gamma = \frac{\kappa}{\sqrt{1+q^2}} \frac{L}{4}$ ; at  $\tau = \frac{L}{8}$ , the string is a rectangle, with its four vertices being  $(\gamma, \phi) = (\pm \frac{\kappa}{\sqrt{1+q^2}} \frac{L}{8}, \pm q \frac{\kappa}{\sqrt{1+q^2}} \frac{L}{8})$ ; at  $\tau = \frac{L}{4}$ , the string is folded and lies on the  $\phi$ -axis, extending from  $\phi = -q \frac{\kappa}{\sqrt{1+q^2}} \frac{L}{4}$  to  $\phi = q \frac{\kappa}{\sqrt{1+q^2}} \frac{L}{4}$ . See figure 1 (Plots made using Mathematica®). When we take the special case of  $q = 0$ , we get a folded string pulsating on the  $\gamma$ -axis; when we take  $q \rightarrow \infty$ , we get a folded string pulsating on the  $\phi$ -axis.

We can modify the solution by introducing a constant “phase shift”  $\Delta$ , i.e. taking  $\Gamma'(\xi^+) = f(\xi^+)$  and  $\tilde{\Gamma}'(\xi^-) = f(\xi^- + \Delta)$  with  $f(\xi)$  the same as before. As can be seen, the



**Figure 2:** The string solution for  $n = 2, q = 1$ , at  $\tau = 0$ (left figure) and  $\tau = 0.1L$ (right figure); the axes(horizontal axis:  $\phi$ ; vertical axis:  $\gamma$ ) are labeled in units of  $\kappa L$

solution obtained this way is related to the old solution via

$$\gamma_{\text{new}}(\tau, \sigma) = \gamma_{\text{old}}(\tau + \Delta, \sigma - \Delta), \quad \phi_{\text{new}}(\tau, \sigma) = \phi_{\text{old}}(\tau + \Delta, \sigma - \Delta) \quad (3.73)$$

which means that adding the phase shift merely corresponds to a shift in  $\tau$ .

**Example II.** Let us generalize the above example by introducing an additional parameter  $n$ . Take

$$\Gamma'(\xi^+) = f(\xi^+), \quad \tilde{\Gamma}'(\xi^-) = f(n\xi^-) \quad (3.74)$$

with  $n$  being a positive integer, and  $f(\xi)$  the same as before. This leads to

$$\gamma = F(\xi^+) + \frac{1}{n}F(n\xi^-), \quad \phi = q \left[ F(\xi^+) - \frac{1}{n}F(n\xi^-) \right] \quad (3.75)$$

Taking the special case of  $n = 1$  we get back example I. For  $n > 1$ , in general we get self-crossing strings. See figure 2 for the  $n = 2$  case. One thing worth pointing out is, when  $n > 1$  the string never shrinks to a point, not even in the special cases of  $q = 0, \infty$  (while when  $n = 1$  the string shrinks to a point at  $\tau = \frac{L}{4}$  for  $q = 0$ , and at  $\tau = 0$  for  $q = \infty$ ), due to the  $\frac{1}{n}$  factor in front of the right moving part  $F(n\xi^-)$ .

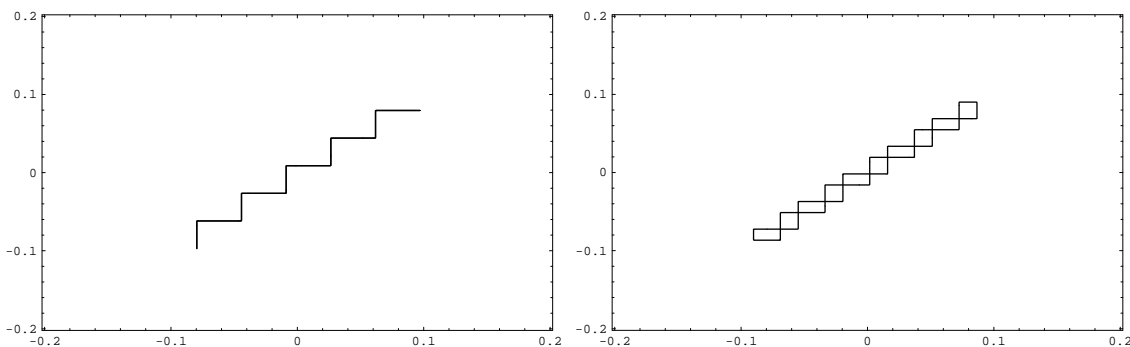
What is quite interesting is the large  $n$  limit. In this limit,  $\gamma \approx F(\xi^+)$ ,  $\phi \approx qF(\xi^+)$ , and the string looks roughly like a static straight line extending from

$$\left( \gamma = -\frac{\kappa}{\sqrt{1+q^2}} \frac{L}{8}, \phi = -q \frac{\kappa}{\sqrt{1+q^2}} \frac{L}{8} \right)$$

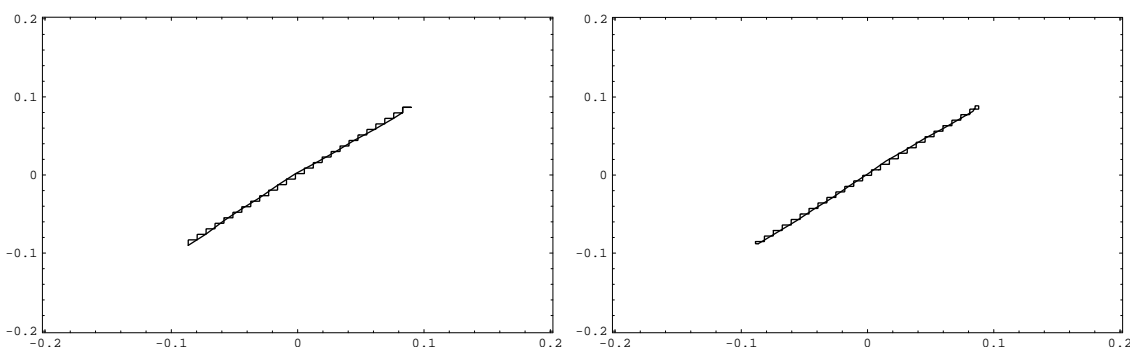
and

$$\left( \gamma = \frac{\kappa}{\sqrt{1+q^2}} \frac{L}{8}, \phi = q \frac{\kappa}{\sqrt{1+q^2}} \frac{L}{8} \right).$$

However, when zooming in, one sees a lot of small rectangles oscillating very rapidly (with period  $\delta\tau \sim \frac{L}{2n}$ ), which come from the right moving part  $\frac{1}{n}F(n\xi^-)$ . These fast oscillating tiny kinks account for the acceleration caused by the string's tension and prevent the string from contracting. See figure 3 and figure 4.



**Figure 3:** The string solution for  $n = 10, q = 1$ , at  $\tau = 0$ (left figure) and  $\tau = 0.035L$ (right figure); the axes(horizontal axis:  $\phi$ ; vertical axis:  $\gamma$ ) are labeled in units of  $\kappa L$



**Figure 4:** The string solution for  $n = 50, q = 1$ , at  $\tau = 0$ (left figure) and  $\tau = 0.003L$ (right figure); the axes(horizontal axis:  $\phi$ ; vertical axis:  $\gamma$ ) are labeled in units of  $\kappa L$

#### 4. Comparison with the “2d-dual” version of NR system

In this section, we shall compare our solutions given in section 3.1 with the “2d-dual” of Neumann-Rosochatius(NR) system and show their differences. As explained in [44, 9], interchanging the roles of the worldsheet coordinates  $\tau$  and  $\sigma$  (the “2d-dual” version) turns rigidly rotating strings into pulsating strings. Let us briefly review this material below.

The rotating string ansatz on  $S^5$  is

$$X_1 + iX_2 = z_1(\sigma)e^{i\omega_1\tau}, \quad X_3 + iX_4 = z_2(\sigma)e^{i\omega_2\tau}, \quad X_5 + iX_6 = z_3(\sigma)e^{i\omega_3\tau} \quad (4.1)$$

with  $z_i(\sigma) = r_i(\sigma)e^{i\alpha_i(\sigma)}, i = 1, 2, 3$ . Interchanging  $\tau \leftrightarrow \sigma$  we get the pulsating string ansatz

$$X_1 + iX_2 = z_1(\tau)e^{im_1\sigma}, \quad X_3 + iX_4 = z_2(\tau)e^{im_2\sigma}, \quad X_5 + iX_6 = z_3(\tau)e^{im_3\sigma} \quad (4.2)$$

with  $z_i(\tau) = r_i(\tau)e^{i\alpha_i(\tau)}, i = 1, 2, 3$ . Periodicity of the  $X$ 's implies that the  $m_i$ 's must be integers. Plugging this ansatz into eq. (3.1), we find the unit modulus condition becomes

$$\sum_{i=1}^3 r_i^2 = 1 \quad (4.3)$$

the conformal gauge conditions become

$$\begin{aligned} \sum_{i=1}^3 (\dot{r}_i^2 + r_i^2 \dot{\alpha}_i^2 + r_i^2 m_i^2) &= \kappa^2 \\ \sum_{i=1}^3 m_i r_i^2 \dot{\alpha}_i &= 0 \end{aligned} \tag{4.4}$$

and the equations of motion become

$$\begin{aligned} \ddot{r}_i - r_i \dot{\alpha}_i^2 + (m_i^2 - \Lambda) r_i &= 0 \\ r_i \ddot{\alpha}_i + 2\dot{r}_i \dot{\alpha}_i &= 0 \end{aligned} \tag{4.5}$$

with  $-\Lambda = \sum_{i=1}^3 (\dot{r}_i^2 + r_i^2 \dot{\alpha}_i^2 - r_i^2 m_i^2)$ . Integrating the above equation of motion for  $\alpha_i$  once we get three constants of motion (which are angular momenta)

$$r_i^2 \dot{\alpha}_i = \mathcal{J}_i = \text{const} \tag{4.6}$$

The resulting class of pulsating strings are described by the effective particle-mechanics Lagrangian of the NR integrable system and can be studied in details, for example, one can take the oscillation “level number”  $N$  to be large and consider the  $\frac{\lambda}{N^2}$  expansion of the string’s energy.

The above pulsating string ansatz only gives a subset of all the solutions on  $S^5$ , which is due to the fact that it has a relatively simple  $\sigma$ -dependence ( $e^{im_i\sigma}$ ) and its  $\tau$ -dependence and  $\sigma$ -dependence are “separated”. In particular, the solutions constructed in section 3.1 depend on both left and right moving coordinates  $\tau \pm \sigma$  in a crucial manner and hence are not included in the NR system pulsating string ansatz. We now illustrate this point by focusing on solutions lying completely in the  $X_1 - X_2$  plane. For the NR system, this corresponds to setting  $r_2 = 0, r_3 = 0$ , which leads to  $r_1 = 1$ . The conformal gauge conditions become

$$\dot{\alpha}_1^2 + m_1^2 = \kappa^2, \quad m_1 \dot{\alpha}_1 = 0 \tag{4.7}$$

The equations of motion for the  $\alpha_i$ ’s give

$$\mathcal{J}_1 = \dot{\alpha}_1, \quad \mathcal{J}_2 = 0, \quad \mathcal{J}_3 = 0 \tag{4.8}$$

while the equations of motion for the  $r_i$ ’s are automatically satisfied. Hence there are only two possible solutions depending on what value the angular momentum  $\mathcal{J}_1$  takes. If we choose  $\mathcal{J}_1 \neq 0$ , i.e.  $\dot{\alpha}_1 \neq 0$ , then the second equation of the conformal gauge conditions gives  $m_1 = 0$  and we get

$$X_1 + iX_2 = e^{i\alpha_1} = e^{i(\mathcal{J}_1\tau + \alpha_1(0))} \tag{4.9}$$

which is a point-like string orbiting on the  $S^1$  in the  $X_1 - X_2$  plane, with an energy (using the first equation of the conformal gauge conditions)  $\frac{E}{\sqrt{\lambda}} = \kappa = |\mathcal{J}_1|$ . If we choose  $\mathcal{J}_1 = 0$ ,

i.e.  $\dot{\alpha}_1 = 0$ , then the first equation of conformal gauge conditions gives  $\kappa = |m_1|$  and we get

$$X_1 + iX_2 = e^{i\alpha(0)+m_1\sigma} \tag{4.10}$$

which is a static string wound  $m_1$  times around the  $S^1$  in the  $X_1 - X_2$  plane. Next let us look at the solution given as Example I of section 3.4 with  $q = \infty, n = 1$ . This gives  $\gamma = 0, \phi(\tau, \sigma) = \Phi(\xi^+) + \tilde{\Phi}(\xi^-) \neq 0$ , which is a folded string pulsating on the  $\phi$ -axis with its center at rest and its length (in terms of the range of  $\phi$ ) changing periodically between 0 and  $\kappa \frac{L}{2}$  (see section 3.4). Translating to the embedding Cartesian coordinates this is

$$X_1 + iX_2 = e^{i\phi(\tau, \sigma)}, X_3 + iX_4 = 0, X_5 + iX_6 = 0 \tag{4.11}$$

describing a folded string pulsating on the  $S^1$  in the  $X_1 - X_2$  plane with its length changing periodically. This solution is obviously distinct from the point-like string and the static wound string obtained from the NR system.

## 5. Fluctuation spectrum

Given a classical solution of the string equations of motion it is naturally of interest to study the fluctuations in this background and for other semi-classical strings this has proved possible in some cases e.g. [5, 45]. It is possible to consider fluctuations about the most general kinky string solution that we found above, however the formulae are complicated and the resulting fluctuation action has non-constant coefficients. We will restrict our attention to the simple case of the pulsating folded string and in particular the  $q \rightarrow \infty, n = 1$  case (Example I in section 3.4). Starting from the Lagrangian

$$\mathcal{L} = -\frac{\sqrt{\lambda}}{4\pi} G_{\mu\nu} \partial_\alpha X^\mu \partial^\alpha X^\nu \tag{5.1}$$

in conformal gauge and with the metric (here let us consider the only the time and sphere part)

$$ds^2 = -dt^2 + d\gamma^2 + \cos^2 \gamma d\phi_1^2 + \sin^2 \gamma (d\psi^2 + \cos^2 \psi d\phi_2^2 + \sin^2 \psi d\phi_3^2). \tag{5.2}$$

We define the fields  $\phi = \frac{1}{2}(\phi_1 + \phi_2), \hat{\phi} = \frac{1}{2}(\phi_1 - \phi_2)$  and expand  $t = \kappa\tau + \tilde{t}, \phi = \phi_0 + \tilde{\phi}, \hat{\phi} = \tilde{\hat{\phi}}$  and  $\gamma = \tilde{\gamma}$ . It does not matter what values  $\psi$  and  $\phi_3$  take in the classical solution so we do not expand these fields. We get

$$G_{\mu\nu} \partial_\alpha X^\mu \partial^\alpha X^\nu = \kappa^2 + 2\kappa \partial_\tau \tilde{t} + (\partial \tilde{\phi})^2 + (\partial \tilde{\gamma})^2 + \left(1 - \frac{\tilde{\gamma}^2}{2}\right)^2 (\partial \phi_0 + \tilde{\phi} + \tilde{\hat{\phi}})^2 + \tilde{\gamma}^2 ((\partial \psi)^2 + \sin^2 \psi \partial \phi_3^2) + \tilde{\gamma}^2 \cos^2 \psi (\partial \phi_0)^2. \tag{5.3}$$

where  $(\partial A)^2 = -(\partial_\tau A)^2 + (\partial_\sigma A)^2$ . We now write the fluctuations of the spherical coordinates in terms of Cartesian variables  $x^i$

$$(\partial \tilde{\gamma})^2 + \tilde{\gamma}^2 (\partial \psi^2 + \sin^2 \psi \partial \phi_3^2) = \sum_{i=1}^3 (\partial x^i)^2$$

$$\tilde{\gamma}^2 = \sum_{i=1}^3 (x^i)^2 \tag{5.4}$$

and so quadratic fluctuation action is

$$\mathcal{L} \sim -(\partial\tilde{t})^2 + (\partial\tilde{\phi})^2 + (\partial\tilde{\phi})^2 + \sum_{i=1}^3 (\partial x^i)^2 + (-1 + \cos^2 \psi)(\partial\phi_0)^2 (x^i)^2. \tag{5.5}$$

However for the simple folded string we do not need to introduce the new coordinate  $\phi$  but rather we can simply expand about the solution  $t = \kappa\tau + \tilde{t}$ ,  $\phi_1 = \phi_0 + \tilde{\phi}_1$  and  $\gamma = 0$  with the rest of the coordinates being arbitrary. We thus find for the quadratic fluctuations

$$\mathcal{L} \sim -(\partial\tilde{t})^2 + (\partial\tilde{\gamma})^2 + \left(1 - \frac{\tilde{\gamma}^2}{2}\right)^2 (\partial\phi_0 + \partial\tilde{\phi}_1)^2 + \tilde{\gamma}^2 (\partial\Omega_3)^2. \tag{5.6}$$

Where  $\Omega_3$  represents the angles on a three sphere and as before we introduce Cartesian variable  $(\partial\tilde{\gamma})^2 + \tilde{\gamma}^2 (\partial\Omega_3)^2 = \sum_{i=1}^4 (\partial x^i)^2$ . We now have

$$\mathcal{L} \sim -(\partial\tilde{t})^2 + (\partial\tilde{\phi}_1)^2 + \sum_{i=1}^4 (\partial x^i)^2 - (\partial\phi_0)^2 (x^i)^2. \tag{5.7}$$

Explicitly, we expand about the solution  $\gamma = 0$  and  $\phi_1 = qF(\xi^+) - qF(\xi^-)$  and  $F(\xi) = \int_0^\xi f(\tilde{\xi})d\tilde{\xi}$  with the  $f$ 's as in (3.71) and we take  $q \rightarrow \infty$ . Our solution breaks into different regions given by

<b>Region I</b>	$0 \leq \xi^+ \leq \frac{L}{4}$	$0 \leq \xi^- \leq \frac{L}{4}$	$\phi_1 = \kappa\sigma$
<b>Region II</b>	$0 \leq \xi^+ \leq \frac{L}{4}$	$\frac{L}{4} \leq \xi^- \leq \frac{L}{2}$	$\phi_1 = \kappa(\tau - L/2)$
<b>Region III</b>	$\frac{L}{4} \leq \xi^+ \leq \frac{L}{2}$	$0 \leq \xi^- \leq \frac{L}{4}$	$\phi_1 = -\kappa(\tau - L/2)$
<b>Region IV</b>	$\frac{L}{4} \leq \xi^+ \leq \frac{L}{2}$	$\frac{L}{4} \leq \xi^- \leq \frac{L}{2}$	$\phi_1 = -\kappa\sigma$

(5.8)

Thus the bosonic fluctuations of the  $S^5$  coordinates in region II are described by,

$$L = -\frac{1}{4\pi} \left( \sum_{i=1}^4 (-\partial_\tau x_i^2 + \partial_\sigma x_i^2 + \kappa^2 x_i^2) - \kappa^2 - 2\lambda^{\frac{1}{4}} \kappa \partial_\tau \tilde{\phi}_1 - \partial_\tau \tilde{\phi}_1^2 + \partial_\sigma \tilde{\phi}_1^2 \right) \tag{5.9}$$

where we see that the transverse coordinates have a positive mass term just as is found in the BMN limit, moreover the fluctuations in region III are the same except that the sign of the term linear in  $\partial_\tau \phi_1$  is positive. However in regions I and IV we find the the mass term for the transverse coordinates is negative. For example in region I

$$L = -\frac{1}{4\pi} \left( \sum_{i=1}^4 (-\partial_\tau x_i^2 + \partial_\sigma x_i^2 - \kappa^2 x_i^2) + \kappa^2 + 2\lambda^{\frac{1}{4}} \kappa \partial_\sigma \tilde{\phi}_1 - \partial_\tau \tilde{\phi}_1^2 + \partial_\sigma \tilde{\phi}_1^2 \right) \tag{5.10}$$

with region IV having a different sign on the term linear in  $\partial_\sigma\phi_1$ . The fluctuations on the  $AdS_5$  are straightforward

$$L_{AdS} = -\frac{1}{4\pi} \left( \kappa^2 + 2\kappa\partial_\tau\tilde{t} + \partial_\tau\tilde{t}^2 - \partial_\sigma\tilde{t}^2 + \sum_{i=1}^4 (-\partial_\tau z_i^2 + \partial_\sigma z_i^2 + \kappa^2 z_i^2) \right) \quad (5.11)$$

where we here use the Cartesian coordinates made from fluctuations in the radial direction  $\rho$  and the angular coordinates of the  $S^3$  inside  $AdS_5$ , here the mass term is of course positive. We would like to remove the longitudinal fluctuations and to this end we consider the Virasoro constraints to linear order in fluctuations where we once again find different relations in different regions

$$\begin{aligned} \text{Region I} \quad & \partial_\tau\tilde{t} = \partial_\sigma\tilde{\phi}, \quad \partial_\sigma\tilde{t} = \partial_\tau\tilde{\phi} \\ \text{Region II} \quad & \partial_\tau\tilde{t} = \partial_\tau\tilde{\phi}, \quad \partial_\sigma\tilde{t} = \partial_\sigma\tilde{\phi} \\ \text{Region III} \quad & \partial_\tau\tilde{t} = -\partial_\tau\tilde{\phi}, \quad \partial_\sigma\tilde{t} = -\partial_\sigma\tilde{\phi} \\ \text{Region IV} \quad & \partial_\tau\tilde{t} = -\partial_\sigma\tilde{\phi}, \quad \partial_\sigma\tilde{t} = -\partial_\tau\tilde{\phi}. \end{aligned} \quad (5.12)$$

Thus the fluctuations in the  $\phi$  direction can be related to those in the time direction which we can remove by rescaling the worldsheet time coordinate such that  $t = \kappa\tau$ . Our final result is that we find a simple theory of transverse bosons in a quadratic potential. This potential corresponds to a simple real mass term in certain regions of the worldsheet though in others it corresponds to an imaginary mass. This imaginary mass is suggestive of the fact that our solution may not be stable. We should now proceed to solve the equations of motion for these varying quadratic potentials and explicitly check for tachyonic modes, however this is a little complicated so we will postpone it for future work and instead consider some of the possibilities for string decay.

## 6. Decay of the string

The decay of highly excited strings in flat space has been under investigation for some time, recent works being, for example, [27–31]. Let us first review these flat space results to gain some intuition. In [27] very massive closed strings which are maximum angular momentum eigenstates were considered, and its quantum decay rate was obtained from a one-loop calculation. It was found that the dominant decay channel is when  $M_1, M_2$  (masses of the two outgoing states) are both of the same order as  $M$  (mass of the initial string) and are restricted along a curve  $M_2 = F(M_1)$ , with the decay rate given by  $\Gamma(M_1, M_2) \sim M^{-4}$ . The sub-dominant channel is when one of the outgoing states is massless and the other one is very massive, with  $\Gamma(M_1 \sim M, M_2 = 0) \sim M^{-5}$ . The decay rates of all other channels, such as  $(M_1, M_2)$  being off the curve  $M_2 = F(M_1)$ , or the two outgoing states both being massless, are exponentially suppressed. When all the decay channels were summed up, the lifetime of initial string state was found to be  $\mathcal{T} \sim M$ . The initial highly excited

string with maximum angular momentum can be represented as a spinning folded string, and a semi-classical analysis was carried out in [28] for the splitting of such a string into two strings. The masses  $M_1, M_2$  of the outgoing strings were expressed as functions of  $\sigma_b$ , the point at which the initial string splits, and then  $\sigma_b$  was eliminated to give a relation  $M_2 = \tilde{F}(M_1)$ . Amazingly, it was found that the quantum and semi-classical analysis give the same relation, i.e.,  $F$  and  $\tilde{F}$  are identical.

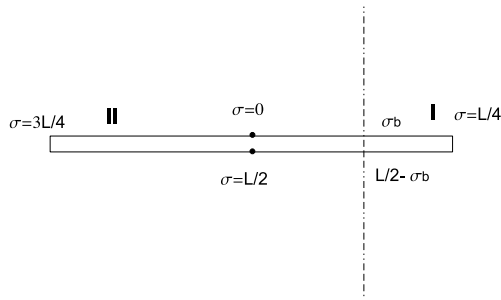
In [29], decay in flat space of very massive string states which are not maximum angular momentum eigenstates was considered. In general, these strings are not folded: they are rotating ellipses and their deformations. Since for these strings there is never a contact between two different points of the string, they will not break classically. Indeed, the one-loop computation of [29] showed that the dominant decay channel is the emitting of massless particles, i.e. one outgoing state being massless while the other outgoing state being massive. All other channels, such as string splitting where both outgoing states are massive, are exponentially suppressed. Among these string states, the special case of a rotating circular string was found to be most stable, with a lifetime  $\mathcal{T} \sim M^5$ . In [30] it was shown that the decay of this rotating circular string can be described by regarding it as a classical radiating antenna. The more recent work [31] constructed flat space string solutions which are rotating and pulsating ellipses that become folded at an instant of time. These strings can break at the moment they become folded. [31] showed that the quantum decay rate of the massive channel for these strings agrees with a semi-classical analysis of their splitting. Other flat space string configurations, such as those having two points that get in contact at an instant of time, and their breaking were also investigated in [31].

It is natural to consider the generalization from flat space to  $AdS_5 \times S^5$ . For example, a similar analysis of the splitting of the folded two-spin string solution of [46] was carried out in [32]. We shall consider the two-parameter  $(q, n)$  family of strings given in section 3.4, which is expected to have various breaking patterns similar to those in flat space. When  $q = 0$  or  $q = \infty$  the string is folded all the time and can break at any moment. When  $q$  is nonzero and finite the string is folded at an instant of time. Furthermore, when  $n > 1$  the string becomes self-intersecting, for example, as depicted in the second plot of figure 2 for the  $n = 2$  case. Below let us first consider the splitting of the string using a semi-classical analysis. As in flat space this should shed light on the massive decay channels of the quantum process.

### 6.1 Splitting of the $n = 1$ string solutions

At any given time, the string can spontaneously split if two points on the string labeled by  $\sigma$  and  $\tilde{\sigma}$  have the same locations in the target space, namely  $\gamma(\tau, \sigma) = \gamma(\tau, \tilde{\sigma}), \phi(\tau, \sigma) = \phi(\tau, \tilde{\sigma})$ . Let us first look at the  $q = \infty$  case where the string is folded along the  $\phi$ -axis. Using the fact that  $f(\xi + \frac{L}{4}) = -f(\xi)$  and  $F(\xi + \frac{L}{4}) = -F(\xi)$  one sees that at any given time, the points labeled by  $\sigma$  and  $\frac{L}{2} - \sigma$  have the same  $\phi$  (and also the same  $\dot{\phi}$ ). Hence let's suppose the string splits at a point labeled by  $\sigma_b$  (and  $\frac{L}{2} - \sigma_b$ ) at time  $\tau_b$ . See figure 5. We compute the charges of the two outgoing pieces I and II by evaluating them at time  $\tau_b$





**Figure 5:** splitting the string ( $n = 1, q = \infty$ )

just before the string splits. The energies are

$$E^I = \sqrt{\lambda} \int_{\sigma_b}^{\frac{L}{2} - \sigma_b} \frac{d\sigma}{L} \kappa = \sqrt{\lambda} \kappa \left( \frac{1}{2} - \frac{2\sigma_b}{L} \right), \quad E^{II} = \sqrt{\lambda} \kappa \left( \frac{1}{2} + \frac{2\sigma_b}{L} \right) \quad (6.1)$$

The angular momenta are

$$J_{12}^I = \sqrt{\lambda} \int_{\sigma_b}^{\frac{L}{2} - \sigma_b} \frac{d\sigma}{L} \frac{1}{2} (\phi_+ + \phi_-) = -2 \frac{\sqrt{\lambda}}{L} q \left[ F \left( \frac{\tau_b + \sigma_b}{2} \right) + F \left( \frac{\tau_b - \sigma_b}{2} \right) \right] \quad (6.2)$$

(where it is understood that the  $q \rightarrow \infty$  limit is taken) and  $J_{12}^{II} = -J_{12}^I$ . All other components  $J_{AB}^{I,II} = 0$ . There is a clear interpretation of this: before splitting, the string pulsates along the  $\phi$  direction, namely, it lies on a circle in the  $X_1 - X_2$  plane with its center of mass at rest and its length changing periodically; after the splitting, one piece goes counter-clockwise and the other goes clockwise on the circle. One remark is in order.  $J_{12}^I$  depends on both when ( $\tau_b$ ) and where ( $\sigma_b$ ) the string splits, hence one cannot get a energy-angular momentum relation  $E^I(J_{12}^I)$  by simply eliminating  $\sigma_b$ . Put another way, the outgoing strings have a continuous spectrum of energy-angular momentum relation parametrized by  $\tau_b$ . This is in contrast to the case considered in [32], where the folded string is a rotating rigid body and hence the energy and angular momenta of the outgoing pieces only depend on where the breaking point is.

The  $q = 0$  case is similar. Now the string is folded along the  $\gamma$ -axis. And at any given time points labeled by  $\sigma$  and  $L - \sigma$  have the same  $\gamma$  and  $\dot{\gamma}$ . Hence we consider the string splitting at the point labeled by  $\sigma_b$  (also  $L - \sigma_b$ ) at time  $\tau_b$ . We find

$$E^I = \sqrt{\lambda} \int_{-\sigma_b}^{\sigma_b} \frac{d\sigma}{L} \kappa = \sqrt{\lambda} \kappa \frac{2\sigma_b}{L}, \quad E^{II} = \sqrt{\lambda} \kappa \left( 1 - \frac{2\sigma_b}{L} \right) \quad (6.3)$$

Also

$$J_{13}^I = 2 \frac{\sqrt{\lambda}}{L} \left[ F \left( \frac{\tau_b + \sigma_b}{2} \right) - F \left( \frac{\tau_b - \sigma_b}{2} \right) \right], \quad J_{13}^{II} = -J_{13}^I \quad (6.4)$$

while all other components  $J_{AB}^{I,II}$  vanish. The interpretation is also similar: before the splitting, the string lies on a circle in the  $X_1 - X_3$  plane with its center of mass at rest and its length changing periodically; after the splitting the two outgoing strings go counter-clockwise and clockwise respectively.

For finite nonzero values of  $q$  the string is the pulsating rectangle, which at  $\tau = 0$  gets folded along the  $\gamma$ -axis, at  $\tau = \frac{L}{4}$  gets folded along the  $\phi$ -axis, and so forth. At  $\tau = 0$ , points at  $\sigma$  and  $L - \sigma$  have the same  $\gamma$ , and the same  $\dot{\gamma}$  (as a matter of fact,  $\dot{\gamma}(\tau = 0, \sigma) = 0$  throughout the string). Their velocities in the  $\phi$ -direction are nevertheless opposite to each other:  $\dot{\phi}(\tau = 0, L - \sigma) = -\dot{\phi}(\tau = 0, \sigma) = -qf(\frac{\sigma}{2})$ . This is because at  $\tau = 0$  the string is starting to re-expand in the  $\phi$ -direction (see figure 1). The energies of the two outgoing pieces are:

$$E^I = \sqrt{\lambda} \kappa \frac{2\sigma_b}{L}, \quad E^{II} = \sqrt{\lambda} \kappa \left(1 - \frac{2\sigma_b}{L}\right) \quad (6.5)$$

while all the angular momenta vanish

$$J_{AB}^I = 0, \quad J_{AB}^{II} = 0 \quad (6.6)$$

since for finite nonzero  $q$ , all of the angular momentum densities are total derivatives of functions of either  $\gamma$  or  $\phi$  w.r.t to  $\sigma$  (see eq. (3.16)). At  $\tau = \frac{L}{4}$  similarly one finds

$$E^I = \sqrt{\lambda} \kappa \left(\frac{1}{2} - \frac{2\sigma_b}{L}\right), \quad E^{II} = \sqrt{\lambda} \kappa \left(\frac{1}{2} + \frac{2\sigma_b}{L}\right) \quad (6.7)$$

and

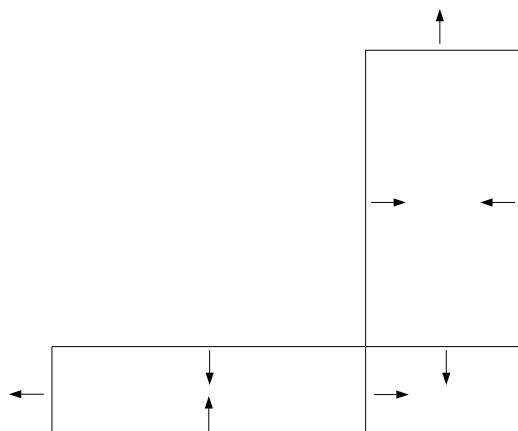
$$J_{AB}^I = 0, \quad J_{AB}^{II} = 0 \quad (6.8)$$

## 6.2 Splitting of the $n > 1$ string solutions

The  $n > 1$  case bears a strong resemblance to the  $n = 1$  case. More specifically, when  $q = \infty$  or  $q = 0$  the string is folded all the time and can break at any time into two pieces each carrying non-vanishing angular momentum  $J_{12}$  or  $J_{13}$ ; when  $q$  is nonzero and finite, the string becomes folded at some instant of time and can break at that time into two pieces both carrying zero angular momentum. However, one new feature that arises in the  $n > 1$  case, for nonzero and finite  $q$ , is its self-crossing and breaking at the self-crossing points.

Let us begin with the  $n = 2$  case (see figure 2). At a generic time  $\tau$  it has a figure “8” shape. The self-crossing point is parametrized by  $\sigma = \frac{L}{4} - \tau$  and  $\tilde{\sigma} = \frac{3L}{4} - \tau$ . Breaking at the self-crossing point gives two outgoing strings with equal energies  $E^I = E^{II} = \frac{\sqrt{\lambda}}{2}\kappa$ , and vanishing angular momenta  $J_{AB}^I = J_{AB}^{II} = 0$ . We have plotted in figure 6 the velocities of the different segments of the string before the breaking. From this, we see that the outgoing strings are two  $n = 1$  strings at different stages of their individual evolution.

Other  $n > 1$  solutions have similar behavior. At a generic time, it has  $n - 1$  nodes (see figure 3 for the  $n = 10$  example), and can break at any of these nodes into an  $(n_1 = n - 1)$ -string and an  $(n_2 = 1)$ -string. The  $(n - 1)$ -string can undergo further splitting in a similar fashion. In this sense, we can say that the  $(n = 1)$ -string is the building block of all the  $n > 1$  strings.



**Figure 6:** The string solution for  $n = 2$  at a generic time  $\tau$ ; velocities of the segments are indicated by arrows; horizontal axis:  $\phi$ ; vertical axis:  $\gamma$

## 7. Discussion

As mentioned previously, kinky strings have been studied in the context of cosmic strings in a flat target space. Additionally, strings with discontinuous worldsheet first derivatives have been studied in some detail in both flat and curved two-dimensional target spacetimes (e.g. [33–36, 47]). It was shown that longitudinal kink modes can be found by carefully taking the massless limit of the massive Nambu string and in  $2d$  flat spacetime, it was shown that the longitudinal kink modes of the folded string could be recast as integrable system of point-like particles connected by linear potentials; the action-angle variables and subsequent canonical quantization of these point-like degrees of freedom were also investigated. These kinky strings may be related to some version of QCD with matter in the adjoint. Interestingly these folded strings have recently appeared in the context of the *AdS/CFT* correspondence [48] where they were considered as the dual description of the excitations of flux tubes connecting quark/anti-quark pairs. The kinky solutions we consider in this paper are parametrized by a constant  $q$ , which in the  $q \rightarrow 0, \infty$  limit become folded strings on a  $S^1 \subset S^5$  and their kinks are exactly analogues of the longitudinal modes of  $2d$  strings in flat spacetime. For generic values of  $q$  our solutions are no longer folded and have transverse modes (they move on an  $S^3 \subset S^5$ ), thus being more complicated. The construction of action-angle variables for the kinks of these solutions and their canonical

quantization definitely deserve more investigation as do their relationship to excitations in the gauge theory.

The kinky solutions considered in this article all have vanishing angular momenta, i.e. the string's center of mass is at rest on the  $S^5$ . One should be able to boost these solutions to give them angular momenta by relaxing the condition that  $\Gamma, \tilde{\Gamma}, \Phi, \tilde{\Phi}$  have to be periodic in their arguments  $\xi^+$  and  $\xi^-$  (see the paragraph under eq. (3.13)). For example, we can boost the  $q = \infty$  folded string solution  $\phi(\tau, \sigma)$  by shifting it to  $\tilde{\phi}(\tau, \sigma) = \phi(\sigma, \tau) + \omega\tau$ , which corresponds to giving linear shifts to the left-moving and right-moving parts  $\Phi(\xi^+) \rightarrow \Phi(\xi^+) + \omega\xi^+$ ,  $\tilde{\Phi}(\xi^-) \rightarrow \tilde{\Phi}(\xi^-) + \omega\xi^-$  thus breaking their periodicity (the string embedding  $\tilde{\phi}(\tau, \sigma)$  is of course still periodic under  $\sigma \rightarrow \sigma + L$ ). One can similarly boost the generic kinky solutions and then take the angular momentum  $J$  to be large. We would like to emphasize that our kinky solutions are different from the “giant magnon” solutions recently constructed in [37] in that the kinky strings are non-rigid pulsating while the giant magnons are rigid strings; furthermore the kinky strings are on  $R_t \times S^3$  while the giant magnons are on  $R_t \times S^2$ . They do share the common feature of being made of straight string segments (although, the kinky strings are in the standard spherical coordinates while the giant magnons are made of straight lines only when cast in LLM coordinates). In fact the giant magnons seem to be closely related to the spiky strings considered in [2] and in particular to those in [3] where the solution of Kruczenski was generalized to a two sphere inside  $S^5$ .

As it turns out, there are more kinky solutions in flat space, which we have not yet been able to generalize to  $R_t \times S^5$ . For example, there is a family of solutions in flat space which we call “pulsating polygonal strings”, describing an  $N$ -sided polygonal string pulsating in the  $(X_1, X_2)$  plane. This solution is obtained by setting

$$\mathbf{a}^{(1)}(u) = \mathbf{f}(u), \quad \mathbf{b}^{(1)}(v) = \mathbf{f}(v) \tag{7.1}$$

with the function  $\mathbf{f}(z)$  given by, written as a complex number  $f(z)$

$$f(z) = \exp \left[ i \frac{2\pi}{N} \text{int} \left( \frac{Nz}{L} \right) \right] \tag{7.2}$$

where  $\text{int}(y)$  is the integer part of  $y$ , e.g.  $\text{int}(0.65) = 1$ ,  $\text{int}(-0.3) = -1$ .

For even values of  $N$ , it is easy to show that at  $\tau = \frac{L}{4}$ , the string shrinks to a point (its center of mass). For odd values of  $N$ , the string never shrinks to a point. For even  $N$  the evolution of the string's profile is: the string starts out as an  $N$ -sided polygon at  $\tau = 0$ ; shrinks to a point at  $\tau = \frac{L}{4}$ ; expands back to the original polygon at  $\tau = \frac{L}{2}$ ; so on and so forth. In the limit of  $N \rightarrow \infty$ , this solution reduces to the known pulsating circular string with its center at  $(X_1 = 0, X_2 = \frac{L}{2\pi})$ .

Can one generalize this family of solutions to strings on  $R_t \times S^5$ ? We can see that the approach in section 3.1 does not generate this desired pulsating polygon solution since the angles between the two string segments joining at the kinks are always 90 degrees in all the solutions obtained there. Let us see whether the NR system discussed in section 4 can do the job. Let us first review a few pulsating solutions that can be obtained from the

NR system. One example is the circular string pulsating on an  $S^2 \subset S^5$  [10]<sup>4</sup>. To get this solution, one takes  $r_3 = 0$ ,  $m_2 = 0$ ,  $\alpha_2 = 0$  (see section 4), so that the string is moving on the  $S^2$  embedded in  $(X_1, X_2, X_3)$ . One also takes  $\mathcal{J}_1 = \mathcal{J}_2 = \mathcal{J}_3 = 0$  so that the string does not have any angular momentum. One also has to take  $\alpha_1 = 0$  so that the  $\alpha_1$  equation of motion is satisfied. Then as can be easily verified, this gives the circular pulsating string, with

$$X_1 + iX_2 = r_1(\tau)e^{im_1\sigma}, \quad X_3 = r_2(\tau), \quad X_4 = X_5 = X_6 = 0 \quad (7.3)$$

with  $r_2 = \sqrt{1 - r_1^2}$  and  $r_1(\tau)$  satisfying

$$\ddot{r}_1 + \left[ \frac{\dot{r}_1^2}{1 - r_1^2} + m_1^2 (1 - r_1^2) \right] r_1 = 0 \quad (7.4)$$

and the energy being

$$\frac{E}{\sqrt{\lambda}} = \kappa = \sqrt{m_1^2 r_1^2 + \frac{\dot{r}_1^2}{1 - r_1^2}} \quad (7.5)$$

In the special case of  $m_1 = 0$ , the above solution reduces to a point like string orbiting in the  $X_1 - X_3$  plane, with  $X_1 = \cos(\omega\tau + \theta_0)$ ,  $X_2 = 0$ ,  $X_3 = \sin(\omega\tau + \theta_0)$ , and  $\frac{E}{\sqrt{\lambda}} = |\omega|$ . More pulsating string solutions from the NR system have also been obtained in [50] for the  $m_2 \neq 0$  case and  $\mathcal{J}_1 \neq 0, \mathcal{J}_2 \neq 0$  case. However, a little thought reveals that the NR system will not give a pulsating polygonal string, because it assumes the  $r_i$ 's to depend only on  $\tau$  but not on  $\sigma$ .

One natural question is what are the dual gauge theory operators of these kinky solutions. As usual one would like to identify a large charge the inverse of which could be used for a power series expansion. Since the kinky solutions have vanishing angular momenta, one should identify a large oscillation quantum number in a way similar to the case of circular pulsating strings [10], which would require a better understanding of the point-like nature of the kink degrees of freedom, preferably at the level of the understanding in the  $2d$  string context described at the beginning of this section. The dual gauge theory operators should be easier to investigate after the solutions are boosted and the large angular momentum limit is then taken. We hope to have something more concrete to say regarding these issues in a future work.

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<sup>4</sup>Similar pulsating circular string solutions were considered in [49]

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